Using Gradual Numbers to Analyze
Non-Monotonic Functions of Fuzzy Intervals

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Abstract

Gradual numbers have been introduced recently as a means of extending standard interval computation methods to fuzzy intervals. The literature treats monotonic functions of fuzzy intervals. In this paper, we combine the concepts of gradual numbers and optimization, which allows for the evaluation of any differentiable function on fuzzy intervals, with no monotonicity requirement.

1   Introduction

The evaluation of a function on a fuzzy interval has long been accomplished by evaluating the function on each α-cut of the the fuzzy interval. Fortin, Dubois, and Fargier recently introduced the concept of gradual numbers [2, 1]. A fuzzy interval can be represented as an interval of gradual numbers, which allows the extension of standard interval computation methods to fuzzy intervals.

Previously published results have extended the practice of analyzing monotonic functions on intervals by evaluating the interval extreme points [2]. In the present paper, we employ optimization techniques on gradual numbers to evaluate differentiable functions on fuzzy intervals with no monotonicity requirements.

2   Classical Interval Computation

Classical interval analysis can be viewed as seeking upper and lower bounds for the value of a operator on interval-valued arguments. For example, when the operator is a real-valued function, $f : \mathbf{R}^n \mapsto \mathbf{R}$ and $x_1 \in [x_1^-, x_1^+], ..., x_n \in [x_n^-, x_n^+]$, the goal is to find z^- and z^+ such that $z = f(x_1, ..., x_n)$ lies in the interval $[z^-, z^+]$ for every $x \in X = \times_i [x_i^-, x_i^+]$.

Warmus [9], Sunaga [8], and Moore [6] first proposed what Lodwick [4] calls an axiomatic approach to interval arithmetic, which defines the following rules for basic interval arithmetic:

For $x \in [x^-, x^+]$ and $y \in [y^-, y^+]$:

1. $x + y = [x^- + y^-, x^+ + y^+]$,

2. $x - y = [x^- - y^+, x^+ - y^-]$,

3. $x \times y = [\min\{x^- \times y^-, x^- \times y^+, x^+ \times y^-, x^+ \times y^+\}, \max\{x^- \times y^-, x^- \times y^+, x^+ \times y^-, x^+ \times y^+\}]$,

4. $x \div y = [\min\{x^- \div y^-, x^- \div y^+, x^+ \div y^-, x^+ \div y^+\}, \max\{x^- \div y^-, x^- \div y^+, x^+ \div y^-, x^+ \div y^+\}]$, $0 \notin [y^-, y^+]$.

Axiomatic interval arithmetic assumes that all variables are independent, including duplicate copies of the same variable. This can lead to overestimations of the bounds of functions. Consider, for example,

$$f(x) = x - x, \ x \in [0, 1].$$

Using the axiomatic approach yields the result

$$[1, 0] - [1, 0] = [0 - 1, 1 - 0] = [-1, 1].$$

If, however, we simplify the function *before* applying axiomatic interval analysis, we get $f(x) = x - x = 0$, which indicates that $[z^-, z^+] = [-1, 1]$ is a gross overestimation.

Simplifying functions to reduce multiple copies of a variable can eliminate the overestimation, but is not always possible. For example, there is no way to express $f(x) = xe^x$ with only one copy of the variable x.

When the function in question is monotonic, we can appeal to a well-known proposition to find exact bounds for $[z^-, z^+]$ without overestimation. Before stating this result, we provide some preliminary definitions. A *configuration*, ω, is an n-tuple which lies in $X = \times_i [x_i^-, x_i^+]$. The set of *extreme configurations*, $\mathcal{H}$, are n-tuples in which every element takes on the value of an interval endpoint. Of course, there are 2^n elements in $\mathcal{H}$. Now consider the following proposition, which indicates what is known as the vertex method of interval analysis in the fuzzy set community.

Proposition 1: Let $x = (x_1, ..., x_n)$ be a tuple of n real interval-valued variables such that $x_i \in [x_i^-, x_i^+]$, and let $z = f(x_1, ..., x_n) = [z^-, z^+]$. If f is continuous and locally monotonic with respect to each argument, then

$$z^- = \min_{\omega \in \mathcal{H}} f(\omega)$$

and

$$z^+ = \max_{\omega \in \mathcal{H}} f(\omega).$$

This follows from the fact that $X = \times_i [x_i^-, x_i^+]$ is a closed subset of R^n, and from the Weierstrass theorem that a continuous function attains its maximum and minimum on a compact set. The vertex method, while effective, is limited in scope. We are still left with over-estimations or underestimations on non-monotonic functions. Take, for example,

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$$f(x) = x(1-x) \ \ for \ x \in [0,1]. \tag{1}$$

If we consider the two instantiations of x to be independent, $\mathcal{H}$ contains the four elements $\{(0,0),(0,1),(1,0),(1,1)\}$, and the result is

$$z^- = f(\omega)^- = \min_{\omega \in \mathcal{H}} f(\omega) = f(1,1) = f(0,0) = f(0,1) = 0,$$

and

$$z^+ = f(\omega)^+ = \max_{\omega \in \mathcal{H}} f(\omega) = f(1,0) = 1.$$

Inspection reveals that there is no single value of x in the interval $[0,1]$ for which $f(x) = x(x-1)$ has a value of 1, so $[z^-, z^+] = [0,1]$ is an unsatisfactory solution. We might, on the other hand, consider the two instantiations of x to be the same variable. Then $\mathcal{H}$ contains only the two elements $\{(0,0),(1,1)\}$, and the result is

$$z^- = f(\omega)^- = \min_{\omega \in \mathcal{H}} f(\omega) = f(1,1) = f(0,0) = 0,$$

and

$$z^+ = f(\omega)^+ = \max_{\omega \in \mathcal{H}} f(\omega) = f(1,1) = f(0,0) = 0.$$

But $[z^-, z^+] = [0,0]$ is also wrong because zero is not the only possible value of f. Consider, for example, $f(\frac{1}{4}) = \frac{3}{16}$. We require a broader tool than the vertex method for evaluation of non-monotonic functions of interval-valued variables with dependencies.

3 Optimization and Interval Analysis

Observe that the values we seek in interval analysis, $z^- = \min_{x \in X} f(x)$ and $z^+ = \max_{x \in X} f(x)$ are the answers to the optimization problems:

$$\text{Minimize} \quad f(x) \tag{2}$$
$$\text{subject to} \quad x_i \geq x_i^-, \quad \text{for } i = 1,...,n \tag{3}$$
$$\qquad\qquad -x_i \geq -x_i^+, \quad \text{for } i = 1,...,n,$$

and

$$\text{Maximize} \quad f(x) \tag{4}$$
$$\text{subject to} \quad x_i \geq x_i^-, \quad \text{for } i = 1,...,n \tag{5}$$
$$\qquad\qquad -x_i \geq -x_i^+, \quad \text{for } i = 1,...,n.$$

Lodwick proposed this idea in the nineties with his constrained interval analysis ([5]), and more recently Kreinovich ([3]) has applied it to technical problems with partial information.

Viewing the interval analysis problem as a mathematical programming problem puts at our disposal the rich optimization theory of the last century, which leads to the following proposition.

Proposition 2: Let $x = (x_1, ..., x_n)$ be a tuple of n variables such that $x_i \in [x_i^-, x_i^+]$ and let $z = f(x_1, ..., x_n)$ such that f is continuously differentiable with respect to each x_i. Now consider the set of KKT configurations, $\mathcal{K}_f^-$ and $\mathcal{K}_f^+$, to be configurations which satisfy the Karush-Kuhn-Tucker conditions for (2)-(3) and (4)-(5) respectively. Then

$$z^- = \min_{\omega \in \mathcal{K}^-} f(\omega)$$

and

$$z^+ = \max_{\omega \in \mathcal{K}+} f(\omega).$$

Proof: Finding the minimum function value for $f(x_1, ..., x_n)$ where $x \in X = \times_i [x_i^-, x_i^+]$ can be stated as a mathematical programming problem:

$$\text{Minimize } z \quad : \quad f(x) \tag{6}$$
$$\text{subject to } c(i) \quad : \quad x_i - x_i^- \geq 0, \quad \text{for } i = 1, ..., n \tag{7}$$
$$c(n+i) \quad : \quad x_i^+ - x_i \geq 0, \quad \text{for } i = 1, ..., n, \tag{8}$$

where x is a $1 \times n$ vector.

Since constraints $c(i)$ and $c(n+i)$ are lower and upper bounds for x_i, at most one at a time of $c(i)$ and $c(n+i)$ will be active. We observe that our mathematical programming problem satisfies the linear independence constraint qualifications (i.e. the gradients of the active constraints are linearly independent), which means that the KKT conditions are necessary conditions for optimality. The details of Karush-Kuhn-Tucker conditions are found in any standard optimization textbook, for example [7].

Now any optimal point will satisfy the following:

$$\nabla_x f(x) - \lambda_1 (x_i - x_i^-) \quad - \quad ... \tag{9}$$
$$... - \lambda_n (x_n - x_n^-) + \lambda_{n+1} (x_1 - x_1^+) \quad + \quad ... \tag{10}$$
$$... + \lambda_{2n} (x_b - x_n^+) \quad = \quad 0 \tag{11}$$
$$\lambda_i \geq 0, \qquad i = 1, ..., 2n \tag{12}$$
$$x_i - x_i^- \geq 0, \qquad i = 1, ..., n \tag{13}$$
$$-x_i + x_i^+ \geq 0, \qquad i = 1, ..., n \tag{14}$$
$$\lambda_i (x_i - x_i^-) = 0, \qquad i = 1, ..., n \tag{15}$$
$$\lambda_n + i(-x_i + x_i^+) = 0, \qquad i = 1, ..., n. \tag{16}$$

The set of points satisfying (11)-(15) are exactly the points in $\mathcal{K}_f^-$. Note that $X = \times_i [x_i^-, x_i^+]$ is a closed subset of R^n, so is a compact set. The Weierstrass theorem guarantees that a continuous function attains a maximum and minimum on a compact set, so we know $f(x)$ has a minimum on X, which has to be $\min_{x \in \mathcal{K}_f^-} f(x)$.

To find the maximum, we replace $f(x)$ in (6) with $-f(x)$, and find the KKT points, which are exactly $\mathcal{K}_f^+$.

3.1 Examples of Interval Analysis via Optimization

For the sake of illustration, recall example (1):

$$f(x) = x(1-x), \ x \in [0,1]. \tag{17}$$

The Karush-Kuhn-Tucker conditions for (17) are

$$
\begin{aligned}
-2x + 1 - \lambda_1 + \lambda_2 &= 0 & (18) \\
\lambda_i &\geq 0, \ j = 1,2 & (19) \\
x &\geq 0, & (20) \\
1 - x &\geq 0, & (21) \\
\lambda_1 x &= 0 & (22) \\
\lambda_2(1-x) &= 0. & (23)
\end{aligned}
$$

$\mathcal{K}_f^-$ consists of the three points which satisfy (18)-(16):

- $x = 0$; $\lambda = (1,0)$; $f(x) = 0$

- $x = 1$; $\lambda = (0,1)$; $f(x) = 0$

- $x = \frac{1}{2}$; $\lambda = (0,0)$; $f(x) = \frac{1}{4}$.

Now $z^- = \min_{x \in \mathcal{K}_f^-} f(x) = 0$. In this example, it happens that $\{x \in \mathcal{K}_f^+\} = \{x \in \mathcal{K}_f^-\}$, so $z^+ = \max_{x \in \mathcal{K}_f^+} f(x) = \frac{1}{4}$.

Optimization yields the desired result, $[z^-, z^+] = [0, \frac{1}{4}]$ without overestimation or underestimation. It should be noted that this accuracy is not without its cost. The vertex method tells us that, for monotonic functions $f : \mathbf{R}^n \mapsto \mathbf{R}$, function extrema will always occur at interval endpoints, so $|\mathcal{H}| = 2^n$. In contrast, the number of KKT configurations of an interval constrained mathematical programming problems will sometimes be fewer than the number of extreme configurations, and will occasionally be many, many more. In the worst case, the number of points satisfying the KKT conditions will be infinite.

4 Gradual Numbers and Fuzzy Interval Analysis

Up to this point, we have considered intervals of real numbers, which provide a Boolean model for uncertainty. Numbers inside the interval are entirely possible, while numbers outside the interval are impossible. It is often the case, however, that the sets we wish to model are not so crisply defined. The need frequently arises to move gradually from impossible to possible, with values on the outskirts of the interval being somewhat possible. Fuzzy set theory accomplishes this by making the bounds of the interval softer, so that the transition from an impossible value to an entirely possible value is gradual.

A fuzzy interval, $\tilde{W}$ can be defined by a membership function, $\mu_{\tilde{W}} : \mathbf{R} \mapsto (0,1]$, where $\mu_{\tilde{W}}(x)$ indicates to what extent x is a member of $\tilde{W}$. A standard fuzzy interval is

the $L-R$ fuzzy interval, which is defined by a membership function $\mu_{\tilde{W}}$ and reference functions L and R. $L : [0, \inf) \to [0,1]$ and $R : [0, \inf) \to [0,1]$ are upper semi-continuous and strictly decreasing in the range (0,1], and $\mu_{\tilde{W}}$ is defined as follows:

$$\mu_{\tilde{W}}(x) \quad = \quad \begin{cases} 1 & \text{for } x \in [w^-, w^+], \\ L(\frac{w^- - x}{\alpha_W}) & \text{for } x < w^-, \\ R(\frac{x - w^+}{\beta_W}) & \text{for } x > w^+. \end{cases} \tag{24}$$

Fortin, Dubois, and Fargier [2] recently proposed that a fuzzy interval can be represented as a set of gradual numbers that lie between two gradual number endpoints in the same way that a real interval is a set of real numbers that lie between two real endpoints.

Definition 1: A *gradual real number* (or gradual number), $\tilde{r}$, is defined by an assignment function $A_{\tilde{r}} : (0,1] \mapsto \mathbf{R}$. It can be understood as a real value parametrized by $\alpha \in (0,1]$. Then for each α, a real value r_α is given by $A_{\tilde{r}}(\alpha)$. Let G be the set of all gradual real numbers.

We can describe a fuzzy interval $\tilde{W}$ by an ordered pair of two gradual numbers, $[\tilde{w}^-, \tilde{w}^+]$. Since a fuzzy interval is a fuzzy set with an upper-semicontinuous membership function, we require that $A_{\tilde{w}^-}$ be increasing, that $A_{\tilde{w}^+}$ be decreasing, and that $A_{\tilde{w}^-}(1) \leq A_{\tilde{w}^+}(1)$. For our results, we will additionally assume that $A_{\tilde{w}^-}$ and $A_{\tilde{w}^+}$ are continuous and that $A_{\tilde{w}^-}(0)$ and $A_{\tilde{w}^+}(0)$ are defined.

Before considering arithmetic operations on *intervals* of gradual numbers, we need a definition of arithmetic operations on gradual numbers themselves. If $\tilde{r}$ and $\tilde{s}$ are two gradual numbers, then $\tilde{r} + \tilde{s}$ is defined by its membership function $A_{\tilde{r}+\tilde{s}} = A_{\tilde{r}} + A_{\tilde{s}}$. Subtraction, multiplication, division, minimum and maximum are similarly defined. In addition, $\tilde{r}$ is said to be less than $\tilde{s}$ if $A_{\tilde{r}}(\alpha) < A_{\tilde{s}}(\alpha)$ for all $\alpha \in [0,1]$. Greater than and equality are similarly defined. We would like to note that the set of gradual numbers is not a fully ordered set.

Beyond arithmetic operations on gradual numbers, we will find it useful to extend a real-valued function $f(x_1, ..., x_n) : R^n \mapsto R$ to a gradual function, $\dot{f}(\tilde{x}_1, ..., \tilde{x}_n) : G^n \mapsto G$. We define the assignment function of the value of $\dot{f}$ as follows:

$$A_{\dot{f}(\tilde{r}_1, ..., \tilde{r}_n)}(\alpha) \quad = \quad f(\tilde{r}_1(\alpha)..., \tilde{r}_n(\alpha)), \tag{25}$$
$$\forall \alpha \in [0,1] \tag{26}$$

A fuzzy interval $\tilde{W} = [\tilde{w}^-, \tilde{w}^+]$ is the set of all gradual numbers, $\tilde{w}$, for which $\tilde{w}^- \leq \tilde{w} \leq \tilde{w}^+$. If $\tilde{W}$ is an L-R fuzzy interval, then $\tilde{w}^-$ and $\tilde{w}^+$ are defined by the following assignment functions:

$$A_{\tilde{W}^-}(\alpha) \quad = \quad w^- - L^{-1}(\alpha)\alpha_W \tag{27}$$
$$A_{\tilde{W}^+}(\alpha) \quad = \quad w^+ + R^{-1}(\alpha)\beta_W, \tag{28}$$

where L^{-1} (respectively R^{-1}) is the inverse of L (respectively R) in the part of its domain where it is positive.

4.1 Earlier Results on Applying Interval Analysis to Fuzzy Intervals

The representation of a fuzzy interval as a crisp interval of gradual numbers allows us to use tools previously developed for real interval analysis in the analysis of fuzzy intervals.

Fortin, Dubois and Fargier extend the vertex method of interval analysis to intervals of gradual numbers [2]. They define a fuzzy configuration, Ω, which is a tuple of gradual numbers, $(\tilde{r}_1, ..., \tilde{r}_n)$ as follows:

Let fuzzy intervals $\tilde{x}_1...\tilde{x}_n$ be described by their gradual endpoints, $[\tilde{x}_1^-, \tilde{x}_1^+]...[\tilde{x}_n^-, \tilde{x}_n^+]$. A *configuration*, Ω, is an n-tuple of gradual numbers which lies in $\tilde{X} = \times_i [\tilde{x}_i^-, \tilde{x}_i^+]$. The set of *extreme fuzzy configurations*, $\tilde{\mathcal{H}}$, are n-tuples in which every element takes on the value of an interval endpoint. A fuzzy configuration or extreme configuration evaluated at α is defined as the tuple of each of its gradual components evaluated at α. That is $\Omega(\alpha) = (\tilde{r}_1(\alpha), ..., \tilde{r}_n(\alpha))$. We now turn to the vertex method for fuzzy interval analysis.

Proposition 3: Let $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)$ be a tuple of n fuzzy interval-valued variables such that $\tilde{x}_i \in [\tilde{x}_i^-, \tilde{x}_i^+]$. Let $\dot{f} : G^n \mapsto G$ be the gradual extension of $f : \mathbf{R}^n \mapsto \mathbf{R}$ as defined in (25), and define the interval $[\tilde{z}^-, \tilde{z}^+]$ to be the smallest interval of gradual numbers that contains $\tilde{z} = \{\dot{f}(\tilde{x}_1, ..., \tilde{x}_n)\}$ for every $\tilde{x} \in \tilde{X} = \times_i [\tilde{x}_i^-, \tilde{x}_i^+]$. If f is continuous and locally monotonic with respect to each argument, then

$$\tilde{z}^- = \dot{\min}_{\Omega \in \tilde{\mathcal{H}}} \dot{f}(\Omega),$$

and

$$\tilde{z}^+ = \dot{\max}_{\Omega \in \tilde{\mathcal{H}}} \dot{f}(\Omega).$$

The reader is referred to [2] for the proof.

An example of the vertex method for intervals of gradual numbers will be instructive. Let $\tilde{x} = [\tilde{x}^-, \tilde{x}^+]$ be a fuzzy interval defined by the following gradual endpoints:

$$\tilde{x}^-(\alpha) = \frac{1}{2}\alpha, \; \alpha \in [0,1] \tag{29}$$

$$\tilde{x}^+(\alpha) = 1 - \frac{1}{2}\alpha, \; \alpha \in [0,1], \tag{30}$$

$$\tag{31}$$

as in Figure 1.

Let $\dot{f} = \tilde{x}^2$ for $\tilde{x} \in \tilde{X}$. To apply the vertex method, we determine that there are two elements in $\tilde{\mathcal{H}}$: $\tilde{x}^-(\alpha) = \frac{1}{2}\alpha, \; \alpha \in [0,1]$ and $\tilde{x}^+(\alpha) = 1 - \frac{1}{2}\alpha, \; \alpha \in [0,1]$. Now

$$\dot{f}(\tilde{x}^-(\alpha)) = \frac{1}{4}\alpha^2, \; \alpha \in [0,1] \tag{32}$$

$$\dot{f}(\tilde{x}^+(\alpha)) = (1 - \frac{1}{2}\alpha)^2 \tag{33}$$

$$= 1 - \alpha + \frac{1}{4}\alpha^2, \; \alpha \in [0,1]. \tag{34}$$
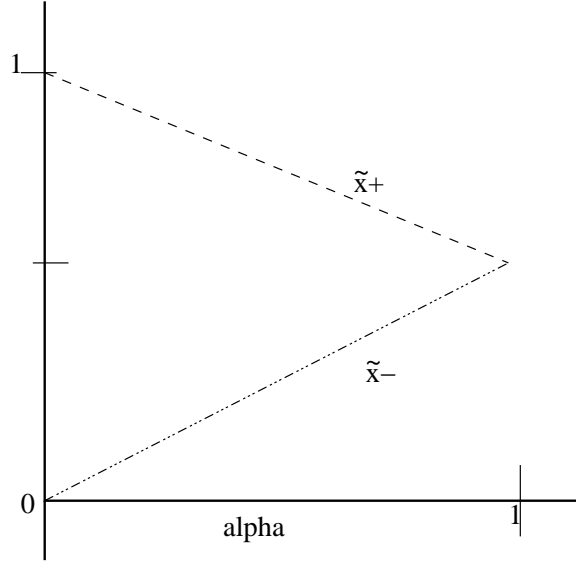
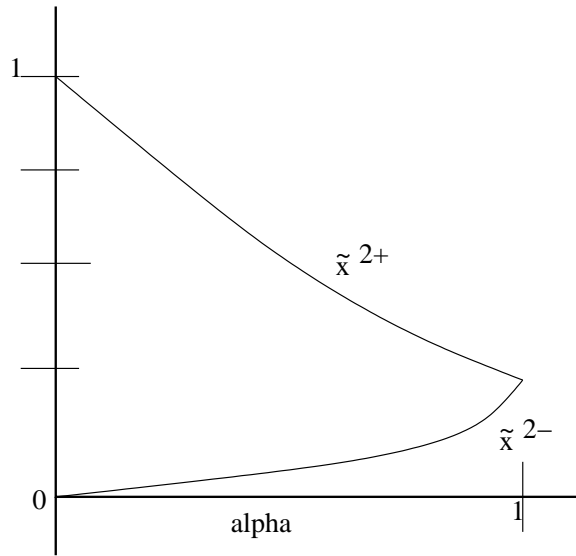Figure 1: $\tilde{x} = [\tilde{x}^-, \tilde{x}^+] = [\frac{1}{2}\alpha, 1 - \frac{1}{2}\alpha]$.

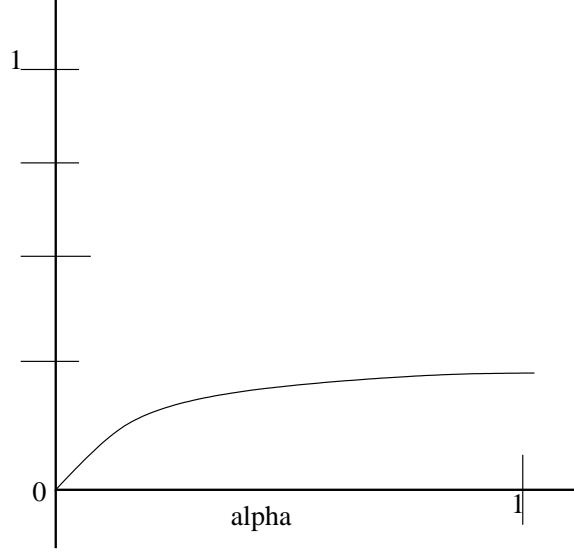Figure 2: $\tilde{x}^2 = [\tilde{z}^-, \tilde{z}^+] = [\frac{1}{4}\alpha^2, 1 - \alpha + \frac{1}{4}\alpha^2]$.

Figure 3: $\tilde{z}^- = \tilde{z}^+ = \frac{1}{2}\alpha - \frac{1}{4}(\alpha)^2$.

It is clear that $\dot{f}(\tilde{x}^-(\alpha)) \leq \dot{f}(\tilde{x}^+(\alpha))$ for all $\alpha \in [0,1]$, so $z^- = \dot{f}(\tilde{x}^-)$ and $z^+ = \dot{f}(\tilde{x}^+)$. The fuzzy interval $\tilde{z} = [\tilde{z}^-, \tilde{z}^+]$ is shown in Figure 2.

The vertex method, while effective, is limited in scope. We are still left with over-estimations, underestimations, or meaningless results on non-monotonic functions. For example, let $\tilde{X}$ be defined as above, and let $\dot{f}$ be the gradual extension of $f(x) = x(1-x)$ for $\tilde{x} \in \tilde{X}$. To apply the vertex method, we determine that there are two elements in $\tilde{\mathcal{H}}$: $\tilde{x}^-(\alpha) = \frac{1}{2}\alpha, \ \alpha \in [0,1]$ and $\tilde{x}^+(\alpha) = 1 - \frac{1}{2}\alpha, \ \alpha \in [0,1]$. Now

$$\dot{f}(\tilde{x}^-(\alpha)) \ = \ \frac{1}{2}\alpha(1 - \frac{1}{2}\alpha), \ \alpha \in [0,1] \tag{35}$$

$$= \ \frac{1}{2}\alpha - \frac{1}{4}\alpha^2, \ \alpha \in [0,1] \tag{36}$$

$$= \ \dot{f}(\tilde{x}^+(\alpha)). \tag{37}$$

Then $\tilde{z}^- = \tilde{z}^+$, so the fuzzy interval is not an interval at all, but a single gradual number defined by the assignment function $\mathcal{A}_{\tilde{z}}(\alpha) = \frac{1}{2}\alpha - \frac{1}{4}(\alpha)^2$, as in Figure 3.

5 Main Results

Fortin, Dubois, and Fargier have used the concept of gradual numbers to extend interval techniques to the evaluation of monotonic functions of fuzzy intervals. In this section we expand the use of gradual numbers. We employ optimization theory in conjunction with gradual numbers to evaluate continuously differentiable non-monotonic functions of fuzzy-interval valued variables.

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5.1 Optimization on Fuzzy Intervals

Let $\dot{f} : G^n \mapsto G$ be the gradual extension of a continuously differentiable function $f : \mathbf{R}^n \mapsto \mathbf{R}$ as defined in (25), and define the interval $[\tilde{z}^-, \tilde{z}^+]$ to be the smallest interval of gradual number that contains $\tilde{z} = \{\dot{f}(\tilde{x}_1, ..., \tilde{x}_n)\}$ for every $\tilde{x} \in \tilde{X} = \times_i [\tilde{x}_i^-, \tilde{x}_i^+]$.

Observe that the values we seek, $\tilde{z}^- = \dot{\min}_{\tilde{x} \in \tilde{\Omega}} \dot{f}(x)$ and $\tilde{z}^+ = \dot{\max}_{\tilde{x} \in \tilde{\Omega}} \dot{f}(\tilde{x})$, are the answers to the optimization problems

$$\text{Minimize} \quad \dot{f}(\tilde{x}) \tag{38}$$
$$\text{subject to} \quad \tilde{x}_i \quad \geq \tilde{x}_i^-, \quad \text{for } i = 1, ..., n \tag{39}$$
$$\tilde{x}_i \quad \leq \tilde{x}_i^+, \quad \text{for } i = 1, ..., n,$$

and

$$\text{Maximize} \quad \dot{f}(\tilde{x}) \tag{40}$$
$$\text{subject to} \quad \tilde{x}_i \quad \geq \tilde{x}_i^-, \quad \text{for } i = 1, ..., n \tag{41}$$
$$\tilde{x}_i \quad \leq \tilde{x}_i^+, \quad \text{for } i = 1, ..., n,$$

We now define the set of KKT configurations, $\tilde{\mathcal{K}}_f^-$ and $\tilde{\mathcal{K}}_f^+$, to be configurations which satisfy the Karush-Kuhn-Tucker conditions for (38)-(39) and (40)-(41) respectively.

Explicitly, $\tilde{\mathcal{K}}_f^+$ is the set of every configuration $x \in \tilde{X}$ which satisfies the following:

$$\dot{g}(\tilde{x}) - \lambda_1(\tilde{x}_i - \tilde{x}_i^-) - ... \tag{42}$$
$$... - \lambda_n(\tilde{x}_n - \tilde{x}_n^-) + \lambda_{n+1}(\tilde{x}_1 - \tilde{x}_1^+) + ... \tag{43}$$
$$... + \lambda_{2n}(\tilde{x}_i - \tilde{x}_n^+) \quad = \quad 0 \tag{44}$$
$$\lambda_i \geq 0, \qquad i = 1, ..., 2n \tag{45}$$
$$\tilde{x}_i - \tilde{x}_i^- \geq 0, \qquad i = 1, ..., n \tag{46}$$
$$-\tilde{x}_i + \tilde{x}_i^+ \geq 0, \qquad i = 1, ..., n \tag{47}$$
$$\lambda_i(\tilde{x}_i - \tilde{x}_i^-) = 0, \qquad i = 1, ..., n \tag{48}$$
$$\lambda_{n+i}(-\tilde{x}_i + \tilde{x}_i^+) = 0, \qquad i = 1, ..., n, \tag{49}$$

where $\dot{g}$ is the gradual extension of $\nabla f(x_1, ..., x_n)$. We define $\tilde{\mathcal{K}}^+(\alpha) = \{\tilde{x}(\alpha) | \tilde{x} \in \tilde{\mathcal{K}}^+$. $\tilde{\mathcal{K}}^-(\alpha)$ is similarly defined.

Proposition 4: Let $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)$ be a tuple of n fuzzy interval-valued variables such that $\tilde{x}_i$ is any gradual number in the interval defined by gradual numbers $[\tilde{x}_i^-, \tilde{x}_i^+]$. Let $[\tilde{z}^-, \tilde{z}^+]$ be the smallest fuzzy interval that contains $\tilde{z} = \dot{f}(\tilde{x}_1, ..., \tilde{x}_n)$ for every $\tilde{x} \in \tilde{X}$, where $\dot{f}$ is the extension of a continuously differentiable real-valued function $f : R^n \mapsto R$. Then

$$\tilde{z}^- = \dot{\min}_{\tilde{\omega} \in \tilde{\mathcal{K}}^-} \dot{f}(\tilde{\omega})$$

and

$$\tilde{z}^+ = \dot{\max}_{\tilde{\omega} \in \tilde{\mathcal{K}}^-} \dot{f}(\tilde{\omega}).$$

Proof: Finding the minimum function value for $\dot{f}(\tilde{x}_1,...,\tilde{x}_n)$ where $\tilde{x} \in \tilde{X} = \times_i [\tilde{x}_i^-, \tilde{x}_i^+]$ can be stated as a mathematical programming problem:

$$\text{Minimize } \tilde{z} \quad : \quad \dot{f}(\tilde{x}) \tag{50}$$

$$\text{subject to } c(i) \quad : \quad \tilde{x}_i - \tilde{x}_i^- \geq 0, \quad \text{for } i = 1,...,n \tag{51}$$

$$c(n+i) \quad : \quad \tilde{x}_i^+ - \tilde{x}_i \geq 0, \quad \text{for } i = 1,...,n, \tag{52}$$

where $\tilde{x}$ is a $1 \times n$ vector of fuzzy interval-valued variables.

Fix $\alpha \in [0,1]$. Since constraints $c(i)$ and $c(n+i)$ are lower and upper bounds for $\tilde{x}_i$ respectively, at most one at a time will be active. We observe that our mathematical programming problem satisfies the linear independence constraint qualifications (the gradients of the active constraints are linearly independent, which means that the KKT conditions are necessary conditions for optimality. Let $g(x) = \nabla_x f(x)$, and let $\dot{g}(\tilde{x})$ be the extension of $g(x)$.

Then for a fixed $\alpha \in [0,1]$, any optimal configuration, $\tilde{x}(\alpha)$ will satisfy the following:

$$\dot{g}(\tilde{x}(\alpha) - \lambda_1(\tilde{x}_i(\alpha) - \tilde{x}_i^-(\alpha)) - ... \tag{53}$$

$$... - \lambda_n(\tilde{x}_n(\alpha) - \tilde{x}_n^-(\alpha)) \tag{54}$$

$$+\lambda_{n+1}(\tilde{x}_1(\alpha) - \tilde{x}_1^+(\alpha)) + ... \tag{55}$$

$$... + \lambda_{2n}(\tilde{x}_i(\alpha) - \tilde{x}_n^+(\alpha)) \quad = \quad 0 \tag{56}$$

$$\lambda_i \geq 0, \quad \text{for} \quad i = 1,...,2n \tag{57}$$

$$\tilde{x}_i(\alpha) - \tilde{x}_i^-(\alpha) \geq 0, \quad i = 1,...,n \tag{58}$$

$$-\tilde{x}_i(\alpha) + \tilde{x}_i^+(\alpha) \geq 0, \quad i = 1,...,n \tag{59}$$

$$\lambda_i(\tilde{x}_i(\alpha) - \tilde{x}_i^-(\alpha)) = 0, \quad i = 1,...,n \tag{60}$$

$$\lambda_{n+i}(-\tilde{x}_i(\alpha) + \tilde{x}_i^+(\alpha)) = 0, \quad i = 1,...,n. \tag{61}$$

Note that since $X(\alpha) = \times_i [\tilde{x}_i^-(\alpha), \tilde{x}_i^+(\alpha)]$ is a closed subset of R^n, it is a compact set. Weierstrass' theorem guarantees that a continuous function attains a maximum and minimum on a compact set, so we know $f(x)$ has a minimum on $X(\alpha)$. which has to be $\min_{\tilde{x} \in \tilde{\mathcal{K}}_f^-} \dot{f}(\tilde{x}(\alpha))$.

The set of points satisfying (56)-(61) for a fixed $\alpha \in [0,1]$ are exactly the points in $\tilde{\mathcal{K}}_f^-(\alpha)$. The minimum of this set will be $\min_{\tilde{x} \in \tilde{\mathcal{K}}_f^-} \dot{f}(\tilde{x}(\alpha))$. Taking this result for every $\alpha \in [0,1]$ allows us to define $\tilde{z}^-$ by its assignment function $\mathcal{A}_{\tilde{z}^-}(\alpha) = \min_{\tilde{x} \in \tilde{\mathcal{K}}_f^-} \dot{f}(\tilde{x}(\alpha))$, which is the same as $\tilde{z}^- = \min_{\tilde{x} \in \tilde{\mathcal{K}}_f^-} \dot{f}(\tilde{x})$.

The proof for $\tilde{z}^+$, is anologous, replacing $\dot{g}(x)$ in (50) with $-\dot{g}(x)$.

5.2 Example of Fuzzy Interval Analysis via Optimization

For the sake of illustration, recall our earlier example:

$$f(x) = x(1-x). \tag{62}$$

Now let $\acute{f}(\tilde{x}) = \tilde{x}(1 - \tilde{x})$ be the gradual extension of $f(x)$, and let $\tilde{x}$ be restricted by the fuzzy interval $\tilde{X} = [\tilde{x}^-, \tilde{x}^+]$, defined by the following gradual endpoints:

$$\tilde{x}^-(\alpha) \;=\; \frac{1}{2}\alpha, \;\; \alpha \in [0,1] \tag{63}$$

$$\tilde{x}^+(\alpha) \;=\; 1 - \frac{1}{2}\alpha, \;\; \alpha \in [0,1]. \tag{64}$$

Finding the right endpoint of $\acute{f}(\tilde{x})$ is equivalent to solving the following optimization problem:

$$\text{Minimize} \qquad \tilde{x} - \tilde{x}^2 \tag{65}$$

$$\text{subject to} \qquad \tilde{x} - \frac{1}{2}\alpha \geq 0 \tag{66}$$

$$\frac{1}{2}\alpha - 1 - \tilde{x} \geq 0 \tag{67}$$

Now $f(x)$ is continuously differentiable, so we can use Proposition 4 to evaluate $\acute{f}(\tilde{x})$ on $\tilde{X}$.

The Karush-Kuhn-Tucker conditions for (17) are

$$1 - 2\tilde{x} - \lambda_1 + \lambda_2 \;=\; 0 \tag{68}$$

$$\lambda_i \;\geq\; 0, \; j = 1,2 \tag{69}$$

$$\tilde{x} - \frac{1}{2}\alpha \;\geq\; 0 \tag{70}$$

$$1 - \frac{1}{2}\alpha - \tilde{x} \;\geq\; 0 \tag{71}$$

$$\lambda_1(\tilde{x} - \frac{1}{2}\alpha) \;=\; 0 \tag{72}$$

$$\lambda_2(1 - \frac{1}{2}\alpha - \tilde{x}) \;=\; 0 \tag{73}$$

$$\lambda_i \;\geq\; 0, \lambda = 1,2 \tag{74}$$

Now $\tilde{\mathcal{K}}_{\acute{f}}^-$ consists of the three triples, $(\tilde{x}, \lambda_1, \lambda_2)$, which satisfy (68)-(74):

1. $(\tilde{x}_a, \lambda_{1a}, \lambda_{2a}) = (\frac{1}{2}\alpha, 1 - \alpha, 0)$

2. $(\tilde{x}_b, \lambda_{1b}, \lambda_{2b}) = (1 - \frac{1}{2}\alpha, 0, 1 - \alpha)$

3. $(\tilde{x}_c, \lambda_{1c}, \lambda_{2c}) = (\frac{1}{2}, 0, 0)$.

We wish to evaluate $\acute{f}$ at all three triples to find the minimizer of $\acute{f}$. Before proceeding, we note that the first two triples in $\tilde{\mathcal{K}}_{\acute{f}}^-$ are defined in terms of α, so we must evaluate feasibility for all $\alpha \in [0,1]$.

1. For $(\tilde{x}_a, \lambda_{1a}, \lambda_{2a})$, all $\alpha \in [0,1]$ are feasible, and $\acute{f}(\tilde{x}_b) = \frac{1}{2}\alpha - \frac{1}{4}\alpha^2, \forall \alpha$.

2. For $(\tilde{x}_b, \lambda_{1b}, \lambda_{2b})$, all $\alpha \in [0,1]$ are feasible, and $\acute{f}(\tilde{x}_b) = \frac{1}{2}\alpha - \frac{1}{4}\alpha^2, \forall \alpha$.

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3. In $(\tilde{x}_c, \lambda_{1c}, \lambda_{2c})$, nothing depends on α and $\acute{f}(\frac{1}{2}) = \frac{1}{4}, \forall \alpha$.

Now $(\tilde{x}_a, \lambda_{1a}, \lambda_{2a}) = (\tilde{x}_b, \lambda_{1b}, \lambda_{2b}) = (1 - \frac{1}{2}\alpha, 0, 1 - \alpha)$ is the minimizer of $\acute{f}$, so the lower endpoint of the fuzzy interval representing $\tilde{x}(1 - \tilde{x})$, is the gradual number $z^- = \frac{1}{2}\alpha - \frac{1}{4}\alpha^2$.

We repeat the steps, minimizing $-\acute{f}(\tilde{x})$, to find z^+, and find that $\tilde{\mathcal{K}}_{\acute{f}}^+$ consists of three triples:

1. $(\tilde{x}_a, \lambda_{1a}, \lambda_{2a}) = (\frac{1}{2}\alpha, \alpha - 1, 0)$

2. $(\tilde{x}_b, \lambda_{1b}, \lambda_{2b}) = (1 - \frac{1}{2}\alpha, 0, \alpha - 1)$

3. $(\tilde{x}_c, \lambda_{1c}, \lambda_{2c}) = (\frac{1}{2}, 0, 0)$.

We wish to evaluate $\acute{f}$ at all three triples to find the minimizer of $\acute{f}$. Before proceeding, we note that the first two triples in $\tilde{\mathcal{K}}_{\acute{f}}^-$ are defined in terms of α, so we must evaluate feasibility for all $\alpha \in (0, 1]$.

1. For $(\tilde{x}_a, \lambda_{1a}, \lambda_{2a})$, only $\alpha = 1$ is feasible, and $\acute{f}(\tilde{x}_a) = \frac{1}{4}$ for $\alpha = 1$ and is undefined otherwise.

2. For $(\tilde{x}_b, \lambda_{1b}, \lambda_{2b})$, only $\alpha = 1$ is feasible, and $\acute{f}(\tilde{x}_b) = \frac{1}{4}$ for $\alpha = 1$ and is undefined otherwise.

3. In $(\tilde{x}_c, \lambda_{1c}, \lambda_{2c})$, nothing depends on α and $\acute{f}(\frac{1}{2}) = \frac{1}{4}$, $\forall \alpha$.

Now clearly $(\tilde{x}_c, \lambda_{1c}, \lambda_{2c}) = (\frac{1}{2}, 0, 0)$ is the maximizer of $\acute{f}$, so the upper endpoint of the fuzzy interval representing $\tilde{x}(1 - \tilde{x})$, is the gradual number $z^- = \frac{1}{4}$, $\forall \alpha \in [0, 1]$.

Our result, the fuzzy interval $[\tilde{z}^-, \tilde{z}^+] = [\frac{1}{2}\alpha - \frac{1}{4}\alpha^2, \frac{1}{4}], \forall \alpha \in (0, 1]$ is illustrated in Figure 4.

6 Conclusion

Viewing a fuzzy interval as an interval of gradual numbers puts at our disposal a number of significant tools previously used only for real interval analysis or for analysis of α-cuts of fuzzy intervals. We have shown that optimization problems with bound constraints can be useful in evaluating non-monotonic functions on fuzzy intervals.

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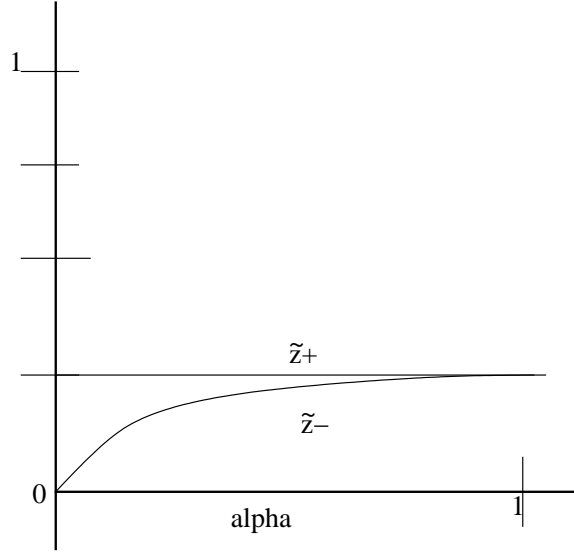
Figure 4: $[\tilde{z}^-, \tilde{z}^+] = [\frac{1}{2}\alpha - \frac{1}{4}\alpha^2, \frac{1}{4}], \forall \alpha \in [0,1]$

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